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THE EFFECT OF REPEATED DIFFERENTIATION ON L-FUNCTIONS

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Abstract. We show that under repeated differentiation, the zeros of the Selberg Ξ-function become more evenly spaced out, but with some scaling towards the origin. We do this by showing the high derivatives of the Ξ-function converge to the cosine function, and this is achieved by expressing a product of Gamma functions as a single Fourier transform.

1. Introduction

In 2006 Haseo Ki [5] proved a conjecture of Farmer and Rhoades [2], that differentiating the Riemann Ξ-function evens out the zero spacing. Specifically Ki showed that there exists sequences A_n and C_n with $C_n \to 0$ slowly such that

$$\lim_{n\to\infty} A_n \Xi^{(2n)}(C_n z) = \cos(z), \tag{1.1}$$

In this paper we extend Ki's result to the entire Selberg Class of L-functions, showing that there exists sequences A_n and C_n (which depend on the properties of L-function under consideration) and constants M' and θ, such that

$$\lim_{n\to\infty} A_n \Xi_F^{(2n)} \left(C_n z - \frac{M'}{\Lambda} \right) = \cos(z+\theta).$$

where Ξ_F is the Xi-function for the L-function F, an element of the Selberg Class. This result is stated more precisely in Theorem 3.1.

In [6], Selberg proposed an axiomatic definition of an L-function, now known as the Selberg Class.

Definition. A function $F(s)$ is an element of the Selberg Class if:

(1) It has a Dirichlet series of the form

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

which is absolutely convergent for $\mathrm{Re}(s) > 1$.

(2) It is a meromorphic function such that $(s-1)^m F(s)$ is an entire function of order 1, for some integer $m \geq 0$.

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(3) It has a functional equation of the form $\Phi(s) = \overline{\Phi(1-\overline{s})}$, where

$$\Phi(s) = \epsilon Q^s F(s) \prod_{j=1}^{k} \Gamma(\lambda_j s + \mu_j)$$

with ϵ, Q, λ_j and μ_j all constants, and subject to $|\epsilon| = 1$, $Q > 0$, $\lambda_j > 0$ and $\mathrm{Re}(\mu_j) \geq 0$.

(4) The coefficients in the Dirichlet series satisfy $a_1 = 1$ and $a_n = O(n^\delta)$ for some fixed positive δ.

(5) It has an Euler product in the sense that

$$\log F(s) = \sum_n \frac{b_n}{n^s}$$

with $b_n = 0$ unless when $n = p^r$ for some prime p and r a positive integer, and $b_n = O(n^\theta)$ for some $\theta < 1/2$.

Kaczorowski and Perelli [4] define an Extended Selberg Class, essentially by dropping the requirement for the function to satisfy an Euler product. Our results apply equally to elements of this extended class of L-functions.

Definition. A function $F(s)$ is an element of the Extended Selberg Class if it satisfies axioms (1)–(3) above.

Remark. The degree of an L-function is 2Λ, where

$$\Lambda = \sum_{j=1}^{k} \lambda_j.$$

It is conjectured that the degree is always an integer. However, this is only known for L-functions of degree 2 or less [4]. More specifically, it is believed that, using the duplication formula, the gamma functions can be transformed so that $\lambda_j = 1/2$ for all j (and in such a case, the L-function has degree k).

Definition. Let F be an element of the Selberg Class, and set

$$\xi_F(s) = s^m (1-s)^m \epsilon Q^s \prod_{j=1}^{k} \Gamma(\lambda_j s + \mu_j) F(s).$$

Note that by assumption of F being in the Selberg Class, $\xi_F(s)$ is an entire function of order 1, with the functional equation $\xi_F(s) = \overline{\xi_F(1-\overline{s})}$.

Definition. Set $\Xi_F(z) = \xi_F(\frac{1}{2} + \mathrm{i}z)$.

Remark. From the functional equation $\Xi_F(z)$ is a real function for $z \in \mathbb{R}$. If the Dirichlet coefficients of F are real, then $\Xi(z)$ is an even function.

Ki proved his result for the Riemann Ξ-function by starting with the integral representation of the Gamma function to show that

$$\Xi_\zeta(z) = \int_{-\infty}^{\infty} \varphi(x) e^{\mathrm{i}xz} \mathrm{d}x,$$

where

$$\varphi(x) = 2 \sum_{n=1}^{\infty} \left(2n^4\pi^2 e^{9x/2} - 3n^2\pi e^{5x/2}\right) e^{-n^2\pi e^{2x}}.$$

Note that the functional equation yields the fact that $\varphi(x) = \varphi(-x)$.

After a suitable change of variables, this yields

$$\Xi_\zeta(z) = 2\pi^2 \int_0^{\infty} e^{-ae^x} e^{bx} \left(1 + O(e^{-x})\right) \left(e^{\mathrm{i}xz/2} + e^{-\mathrm{i}xz/2}\right) \mathrm{d}x,$$

with $a = \pi$ and $b = 9/4$. By differentiating such integrals, Ki was able to explicitly show the existence of sequences A_n and C_n such that (1.1) held. His method also holds for Hecke L-functions, since the functional equation, analogously to the Riemann Xi-function, can be written with a single Gamma function. However, the Selberg Class of L-functions generally includes a product of disparate Gamma functions, which cannot be simplified down to a single one by the multiplication formula of the Gamma function.

In sections 2 and 3, we find the Fourier transform for the analogous Ξ-function for an element of the (extended) Selberg Class of L-functions, showing it can be written as

$$\Xi_F(z) = B \int_{-\infty}^{\infty} \varphi(x) e^{\mathrm{i}\Lambda zx} \mathrm{d}x,$$

where $\varphi(x) = e^{-ae^x} e^{bx} \left(1 + O(e^{-x})\right)$ as $x \to \infty$, and where $\Lambda = \sum \lambda_j$.

In section 3, we start from that result to demonstrate the existence of sequences A_n and C_n such that

$$\lim_{n\to\infty} A_n \Xi_F^{(2n)} \left(C_n z - \frac{M'}{\Lambda}\right) = \cos(z + \theta)$$

where $\theta = \arg(B)$ and $M' = \sum_{j=1}^{k} \operatorname{Im} \mu_j$. We utilize a similar argument to that used by Ki.

The rates of convergence are considered in section 4, demonstrated by numerical examples.

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2. Expressing the Ξ-function as a Fourier transform

Theorem 2.1. *Let F be an element of the Selberg Class, with data m, k, $\varepsilon, Q, \lambda_j$, and μ_j. The Fourier transform of the Xi-function related to F is*

$$\begin{aligned}\widehat{\Xi}_F(x) &= \int_{-\infty}^{\infty} \Xi_F(z) e^{-\mathrm{i}xz} \mathrm{d}z \\ &= \hat{B} \exp\left(-\hat{a} e^{x/\Lambda} + \hat{b} x\right) \left(1 + O\left(e^{-x/\Lambda}\right)\right)\end{aligned}$$

where

$$\hat{a} = \Lambda Q^{-1/\Lambda} \prod_{j=1}^{k} \lambda_j^{-\lambda_j/\Lambda}$$

and

$$\hat{b} = \frac{2m + M + \frac{1}{2}\Lambda}{\Lambda}$$

and

$$\hat{B} = (-1)^m \varepsilon Q^{-(M+2m)/\Lambda} (2\pi)^{(k+1)/2} \Lambda^{2m-1/2} \prod_{j=1}^{k} \lambda_j^{-\frac{1}{2}+\mu_j+\lambda_j(-M-2m)/\Lambda}$$

where

$$\Lambda = \sum_{j=1}^{k} \lambda_j$$

and

$$M = \sum_{j=1}^{k} \mu_j - \tfrac{1}{2}(k-1).$$

Remark. Note that Λ and M are invariant under the Gamma multiplication formulae.

Recall that

$$\begin{aligned}\Xi_F(z) &= \xi_F(\frac{1}{2} + iz) \\ &= \varepsilon Q^{1/2+iz} \left(\tfrac{1}{4} + z^2\right)^m F(\tfrac{1}{2} + \mathrm{i}z) \prod_{j=1}^{k} \Gamma(\mathrm{i}\lambda_j z + \mu_j + \tfrac{1}{2}\lambda_j)\end{aligned}$$

is an entire function. We wish to find its Fourier transform

$$\widehat{\Xi}_F(x) = \int_{-\infty}^{\infty} \Xi_F(z) e^{-\mathrm{i}xz} \mathrm{d}z.$$

Shifting the contour so that $F(s)$ can be represented by its Dirichlet series, swapping the order of summation and integration and shifting the contour back, we find that

$$\widehat{\Xi}_F(x) = \varepsilon Q^{1/2} \sum_{n=1}^{\infty} \frac{a_n}{n^{1/2}} \int_{-\infty}^{\infty} \left(\tfrac{1}{4} + z^2\right)^m \prod_{j=1}^{k} \Gamma(\mathrm{i}\lambda_j z + \mu_j + \tfrac{1}{2}\lambda_j) \left(\frac{ne^x}{Q}\right)^{-\mathrm{i}z} \mathrm{d}z. \quad (2.1)$$

Thus the Fourier transform can be found by convolutions and differentiations of the Fourier transform of the Gamma function.

Theorem 2.2 (Fourier transform of multiple gamma functions)**.** *Let $\lambda_1, \ldots, \lambda_k > 0$ and let $\alpha_1, \ldots, \alpha_k$ be such that their real parts are all positive. Then for large T,*

$$\int_{-\infty}^{\infty} \left(\prod_{j=1}^{k} \Gamma(\alpha_j + \mathrm{i}\lambda_j z) \right) e^{-\mathrm{i}Tz} \mathrm{d}z$$
$$= C_k \exp\left(-\Lambda e^{T/\Lambda} \prod_{j=1}^{k} \lambda_j^{-\lambda_j/\Lambda} + \frac{T(A - (k-1)/2)}{\Lambda} \right) \left(1 + O\left(e^{-T/\Lambda}\right)\right)$$

where $\Lambda = \sum_{j=1}^{k} \lambda_j$ and $A = \sum_{j=1}^{k} \alpha_j$ and

$$C_k = \frac{(2\pi)^{(k+1)/2}}{\sqrt{\Lambda}} \prod_{j=1}^{k} \lambda_j^{-\frac{1}{2} + \alpha_j + \lambda_j\left(\frac{1}{2}(k-1) - A\right)/\Lambda}. \quad (2.2)$$

Remark. Booker stated a similar result in the case when $\lambda_j = 1/2$ for all j, in section 5.2 of [1].

Proof. We prove this theorem by induction. The base case, when $k = 1$ says that for $\lambda > 0$ and $\mathrm{Re}(\alpha) > 0$,

$$\int_{-\infty}^{\infty} \Gamma(\mathrm{i}\lambda z + \alpha) e^{-\mathrm{i}Tz} \mathrm{d}z = \frac{2\pi}{\lambda} \exp\left(-e^{T/\lambda} + T\alpha/\lambda\right). \quad (2.3)$$

This is simply the Fourier transform of one gamma function, a classical result.

With our choice of Fourier constants the convolution theorem is

$$\int_{-\infty}^{\infty} f(z) g(z) e^{-\mathrm{i}Tz} \mathrm{d}z = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f}(x) \widehat{g}(T - x) \mathrm{d}x$$

where $\widehat{f}$ and $\widehat{g}$ are the Fourier transforms of f and g respectively. The Fourier transform of $k+1$ gamma functions will be the convolution of the Fourier transform of k gamma functions with the Fourier transform of one gamma function, both of which are known by the inductive hypothesis. That is,

$$\begin{aligned}\int_{-\infty}^{\infty}&\left(\prod_{j=1}^{k+1}\Gamma(\alpha_j+\mathrm{i}\lambda_j z)\right)e^{-\mathrm{i}Tz}\mathrm{d}z\\&=\frac{C_k}{\lambda_{k+1}}\int_{-\infty}^{\infty}\exp\left(-\Lambda e^{x/\Lambda}\prod_{j=1}^{k}\lambda_j^{-\lambda_j/\Lambda}+\frac{x(A-(k-1)/2)}{\Lambda}\right)\left(1+O\left(e^{-x/\Lambda}\right)\right)\\&\qquad\qquad\times\exp\left(-e^{(T-x)/\lambda_{k+1}}+\frac{(T-x)\alpha_{k+1}}{\lambda_{k+1}}\right)\mathrm{d}x\qquad(2.4)\end{aligned}$$

where we have set $\Lambda=\sum_{j=1}^{k}\lambda_j$ and $A=\sum_{j=1}^{k}\alpha_j$. Later in the proof, we will also set $\Lambda'=\sum_{j=1}^{k+1}\lambda_j$ and $A'=\sum_{j=1}^{k+1}\alpha_j$.

We will asymptotically evaluate this integral. Note that the exponential in the integrand is dominated by

$$-\Lambda e^{x/\Lambda}\prod_{j=1}^{k}\lambda_j^{-\lambda_j/\Lambda}-e^{(T-x)/\lambda_{k+1}}$$

and this has a maximum at $x=x_0$ where x_0 is such that

$$-e^{x_0/\Lambda}\prod_{j=1}^{k}\lambda_j^{-\lambda_j/\Lambda}+\frac{1}{\lambda_{k+1}}e^{(T-x_0)/\lambda_{k+1}}=0$$

that is

$$x_0=\frac{T\Lambda}{\Lambda'}+\frac{\lambda_{k+1}\Lambda}{\Lambda'}\ln\left(\frac{1}{\lambda_{k+1}}\prod_{j=1}^{k}\lambda_j^{\lambda_j/\Lambda}\right)$$

where $\Lambda'=\Lambda+\lambda_{k+1}=\sum_{j=1}^{k+1}\lambda_j$.

Thus, expanding around $x=x_0+\epsilon$ for small ϵ, we have (after a fair amount of straightforward algebraic simplification, and using the identity $\Lambda'=\Lambda+\lambda_{k+1}$)

$$\begin{aligned}-\Lambda e^{x/\Lambda}\prod_{j=1}^{k}\lambda_j^{-\lambda_j/\Lambda}-e^{(T-x)/\lambda_{k+1}}&=-e^{T/\Lambda'}\prod_{j=1}^{k+1}\lambda_j^{-\lambda_j/\Lambda'}\left(\Lambda e^{\epsilon/\Lambda}+\lambda_{k+1}e^{-\epsilon/\lambda_{k+1}}\right)\\&=-\Lambda' e^{T/\Lambda'}\prod_{j=1}^{k+1}\lambda_j^{-\lambda_j/\Lambda'}\left(1+\frac{1}{2\lambda_{k+1}\Lambda}\epsilon^2+B_1\epsilon^3+O(\epsilon^4)\right)\end{aligned}$$

where B_1 is an inconsequential constant that depends upon Λ and λ_{k+1}. (We remark that it is no surprise the coefficient of the ϵ term is zero, as this is the expansion around the maximum of the LHS).

Substituting $x = x_0 + \epsilon$ in the two other terms in the exponent of the integrand in (2.4) and letting $A' = A + \alpha_{k+1} = \sum_{j=1}^{k+1} \alpha_j$ we have

$$\frac{x(A - \frac{1}{2}(k-1))}{\Lambda} + \frac{(T-x)\alpha_{k+1}}{\lambda_{k+1}} = \frac{T(A' - \frac{1}{2}(k-1))}{\Lambda'} \\ + \frac{\lambda_{k+1}(A - \frac{1}{2}(k-1)) - \alpha_{k+1}\Lambda}{\Lambda'} \ln\left(\frac{1}{\lambda_{k+1}} \prod_{j=1}^{k} \lambda_j^{\lambda_j/\Lambda}\right) + B_2\epsilon$$

where $B_2 = \frac{A - \frac{1}{2}(k-1)}{\Lambda} - \frac{\alpha_{k+1}}{\lambda_{k+1}}$ is another inconsequential constant.

Substituting both these expansions back into (2.4) we see that the Fourier transform of the $k+1$ Gamma functions is asymptotically

$$C \exp\left(-\Lambda' e^{T/\Lambda'} \prod_{j=1}^{k+1} \lambda_j^{-\lambda_j/\Lambda'} + \frac{T(A' - \frac{1}{2}(k-1))}{\Lambda'}\right) \\ \times \int \exp\left(-\epsilon^2 \frac{\Lambda'}{2\lambda_{k+1}\Lambda} e^{T/\Lambda'} \prod_{j=1}^{k+1} \lambda_j^{-\lambda_j/\Lambda'} \left(1 + B_1\epsilon + O(\epsilon^2)\right) + B_2\epsilon\right) \mathrm{d}\epsilon$$

where

$$C = \frac{C_k}{\lambda_{k+1}} \left(\frac{1}{\lambda_{k+1}} \prod_{j=1}^{k} \lambda_j^{\lambda_j/\Lambda}\right)^{\frac{\lambda_{k+1}(A - \frac{1}{2}(k-1)) - \alpha_{k+1}\Lambda}{\Lambda'}}. \tag{2.5}$$

We utilise here the normal methods of asymptotic analysis, where the range of the ϵ integral is thought of as being small (so $O(\epsilon)$ terms are small), but $\epsilon^2 e^{T/\Lambda'}$ is large, so the Gaussian integral can be extended to the whole real line with trivially small error. To be concrete, truncate the ϵ integral to be over $\left[-e^{-T/3\Lambda'}, e^{-T/3\Lambda'}\right]$ and let $Q = \frac{\Lambda'}{2\lambda_{k+1}\Lambda} \prod_{j=1}^{k+1} \lambda_j^{-\lambda_j/\Lambda'}$, so we have

$$\int_{-e^{-T/3\Lambda'}}^{e^{-T/3\Lambda'}} e^{-\epsilon^2 Q e^{T/\Lambda'}\left(1 + B_1\epsilon + O(\epsilon^2)\right) + B_2\epsilon} \mathrm{d}\epsilon \\ = \int_{-e^{-T/3\Lambda'}}^{e^{-T/3\Lambda'}} e^{-\epsilon^2 Q e^{T/\Lambda'}} \left(1 - B_1 Q e^{T/\Lambda'} \epsilon^3 + B_2\epsilon + O\left(e^{2T/\Lambda'}\epsilon^6\right)\right) \mathrm{d}\epsilon.$$

We can extend the integral to be over all $\mathbb{R}$ with a tiny error, of size $O\left(e^{-Qe^{T/3\Lambda'}}\right)$. Note that due to the symmetry of the integral, the odd terms in ϵ vanish, and note that the big-O term in the integrand contributes $O\left(e^{-3T/2\Lambda'}\right)$ to the integral.

Therefore, the above integral equals

$$\int_{-\infty}^{\infty} e^{-\epsilon^2 Q e^{T/\Lambda'}} \left(1+O\left(e^{2T/\Lambda'}\epsilon^6\right)\right) \mathrm{d}\epsilon + O\left(e^{-Qe^{T/3\Lambda'}}\right)$$
$$= \sqrt{\frac{\pi}{Q}} e^{-T/2\Lambda'} \left(1+O(e^{-T/\Lambda'})\right).$$

It is easy to see the contribution to (2.4) from outside the range

$$\left[x_0 - e^{-T/3\Lambda'}, x_0 + e^{-T/3\Lambda'}\right]$$

contributes a tiny amount, dominated by the error term above, and so

$$\int_{-\infty}^{\infty} \left(\prod_{j=1}^{k+1} \Gamma(\alpha_j + \mathrm{i}\lambda_j z)\right) e^{-\mathrm{i}Tz} \mathrm{d}z = \sqrt{\frac{2\pi\lambda_{k+1}\Lambda}{\Lambda'}} \prod_{j=1}^{k+1} \lambda_j^{\lambda_j/(2\Lambda')} C \times$$
$$\times \exp\left(-\Lambda' e^{T/\Lambda'} \prod_{j=1}^{k+1} \lambda_j^{-\lambda_j/\Lambda'} + \frac{T(A' - \frac{1}{2}k)}{\Lambda'}\right) \left(1 + O\left(e^{-T/\Lambda'}\right)\right).$$

In order to simplify the constant, recall the definitions of C given in (2.5) and C_k given in (2.2). After some rearranging, we see that

$$\sqrt{\frac{2\pi\lambda_{k+1}\Lambda}{\Lambda'}} \prod_{j=1}^{k+1} \lambda_j^{\lambda_j/(2\Lambda')} C = \frac{(2\pi)^{(k+2)/2}}{\sqrt{\Lambda'}} \prod_{j=1}^{k+1} \lambda_j^{-1/2+\alpha_k+\lambda_j(k/2-A')/\Lambda'}$$
$$= C_{k+1}$$

which is the required form for $k+1$ Gamma functions, thus completing the proof. □

Corollary 2.3. *Let $\lambda_1, \dots, \lambda_k > 0$ and let $\alpha_1, \dots, \alpha_k$ be such that their real parts are all positive. Then for large T,*

$$\int_{-\infty}^{\infty} \left(\tfrac{1}{4} + z^2\right)^m \left(\prod_{j=1}^{k} \Gamma(\alpha_j + \mathrm{i}\lambda_j z)\right) e^{-\mathrm{i}Tz} \mathrm{d}z$$
$$= C_{k,m} \exp\left(-\Lambda e^{T/\Lambda} \prod_{j=1}^{k} \lambda_j^{-\lambda_j/\Lambda} + \frac{T(2m + A - (k-1)/2)}{\Lambda}\right) \left(1 + O\left(e^{-T/\Lambda}\right)\right)$$

where $\Lambda = \sum_{j=1}^k \lambda_j$ and $A = \sum_{j=1}^k \alpha_j$ and

$$C_{k,m} = (-1)^m (2\pi)^{(k+1)/2} \Lambda^{2m-1/2} \prod_{j=1}^{k} \lambda_j^{-\frac{1}{2}+\alpha_j+\lambda_j\left(\frac{1}{2}(k-1)-A-2m\right)/\Lambda}.$$

Proof. The new term $\left(\frac{1}{4}+z^2\right)^m$ requires the first $2m$ derivatives of the RHS to be calculated. The big-O term is differentiable, and note that it dominates all the derivatives other than the $2m^{\text{th}}$ derivative. The result then follows immediately. $\square$

Proof of Theorem 2.1. First note that from the above Corollary, the contribution to (2.1) for the terms with $n > 1$ are exponentially smaller than the error term in $n = 1$ term, for large x. Since $a_1 = 1$ for an element of the Selberg Class, we have that for large x,

$$\widehat{\Xi}_F(x) = (-1)^m \varepsilon Q^{-(M+2m)/\Lambda}(2\pi)^{(k+1)/2}\Lambda^{2m-1/2}\prod_{j=1}^{k}\lambda_j^{-\frac{1}{2}+\mu_j-\lambda_j(M+2m)/\Lambda}$$

$$\times \exp\left(-\Lambda Q^{-1/\Lambda}e^{x/\Lambda}\prod_{j=1}^{k}\lambda_j^{-\lambda_j/\Lambda} + (2m+M+\tfrac{1}{2}\Lambda)\frac{x}{\Lambda}\right)\left(1+O\left(e^{-x/\Lambda}\right)\right),$$

where we have used the Corollary above, with $\alpha_j = \mu_j + \frac{1}{2}\lambda_j$, $T = x - \log Q$ and we set $M = \sum_{j=1}^{k}\mu_j - \frac{1}{2}(k-1)$. This is the theorem, with the constants $\hat{B}$, $\hat{a}$ and $\hat{b}$ given explicitly. $\square$

Remark. The proof made essential use of only the first three assumptions arising from $F(s)$ being an element of the Selberg class. Therefore this result holds for F an element of the Extended Selberg Class (with $\hat{B}$ being trivially changed if $a_1 \neq 1$).

3. The Ξ-function under repeated differentiation

Note that with our choice of constants, the inverse Fourier transform is

$$\Xi_F(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{\Xi}_F(x)e^{\mathrm{i}xz}\mathrm{d}x.$$

Note that the μ_j, part of the data of the L-function F, could be complex. If we define

$$M' = \sum_{j=1}^{k}\operatorname{Im}\mu_j,$$

and rescale z we have

$$\begin{aligned}\Xi_F\left(\frac{z-M'}{\Lambda}\right) &= \frac{\Lambda}{2\pi}\int_{-\infty}^{\infty}\widehat{\Xi}_F(x\Lambda)e^{-\mathrm{i}xM'}e^{\mathrm{i}xz}\mathrm{d}x\\ &= B\int_{-\infty}^{\infty}\varphi(x)e^{ixz}\mathrm{d}x\end{aligned}$$

where by Theorem 2.1

$$\varphi(x) = e^{-ae^x}e^{bx}\left(1+O(e^{-x})\right), \tag{3.1}$$

with

$$a = \Lambda Q^{-1/\Lambda} \prod_{j=1}^{k} \lambda_j^{-\lambda_j/\Lambda}, \tag{3.2}$$

$$b = 2m + \tfrac{1}{2}\Lambda - \tfrac{1}{2}(k-1) + \sum_{j=1}^{k} \operatorname{Re}\mu_j \tag{3.3}$$

and $B = \hat{B}\Lambda/2\pi$. (Note that $a, b \in \mathbb{R}$ and, in the notation of Theorem 2.1, $a = \hat{a}$ and $b = \Lambda\hat{b} - \mathrm{i}M'$).

Theorem 3.1. *Let $\Xi_F(z)$ be the Xi-function for the L-function F, an element of the Selberg Class. Let w_n be defined as the solution to*

$$a w_n e^{w_n} = b w_n + 2n$$

where a and b are given by (3.2) *and* (3.3) *respectively. Then uniformly on compact subsets of $\mathbb{C}$,*

$$\lim_{n\to\infty} A_n \Xi_F^{(2n)}\left(C_n z - \frac{M'}{\Lambda}\right) = \cos(z + \arg(B)),$$

where Λ, M', and B are given in Theorem 2.1, and the sequences A_n and C_n are given by

$$A_n = (-1)^n \exp\left(a e^{w_n} - b w_n\right) \frac{\sqrt{n}}{2|B|\Lambda^{2n} w_n^{2n+1/2}\sqrt{\pi}}$$

and

$$C_n = \frac{1}{\Lambda w_n}.$$

Remark. One can see that, for large n, the w_n defined in the theorem satisfies

$$w_n \sim \log\left(\frac{2n}{a}\right) - \log\log\left(\frac{2n}{a}\right).$$

Proof. From the functional equation for the L-function we have that

$$\Xi_F\left(\frac{z - M'}{\Lambda}\right) = \overline{\Xi_F\left(\frac{\overline{z} - M'}{\Lambda}\right)}$$

so

$$\begin{aligned} B\int_{-\infty}^{\infty} \varphi(x) e^{ixz}\mathrm{d}x &= \overline{B}\int_{-\infty}^{\infty} \varphi(x) e^{-ixz}\mathrm{d}x \\ &= \overline{B}\int_{-\infty}^{\infty} \varphi(-x) e^{ixz}\mathrm{d}x, \end{aligned}$$

and since this holds for any $z \in \mathbb{C}$ we have

$$B\varphi(x) = \overline{B}\varphi(-x).$$

Therefore
$$\Xi_F\left(\frac{z-M'}{\Lambda}\right)=\int_0^\infty \varphi(x)\left(Be^{ixz}+\overline{B}e^{-ixz}\right)\mathrm{d}x. \tag{3.4}$$
We can now consider just the integral
$$f(z)=\int_0^\infty \varphi(x)e^{ixz}\mathrm{d}x$$
as the second integral will behave in much the same way. Differentiating this, we have that
$$f^{(2n)}(z)=(-1)^n\int_0^\infty \varphi(x)x^{2n}e^{ixz}\mathrm{d}x.$$

Haseo Ki [5] proved that uniformly on compact subsets of $\mathbb{C}$,
$$\lim_{n\to\infty}\int_0^\infty v_n\varphi(w_nx)x^{2n}e^{ixz}\mathrm{d}x=e^{iz},$$
where $\varphi(x)$ is of the form given in (3.1), and w_n is defined such that
$$aw_ne^{w_n}=bw_n+2n$$
and
$$v_n=\sqrt{\frac{nw_n}{\pi}}e^{ae^{w_n}}e^{-bw_n}.$$

Therefore, we have that
$$\begin{aligned}f^{(2n)}(z/w_n)&=(-1)^n\int_0^\infty \varphi(x)x^{2n}e^{ixz/w_n}\mathrm{d}x\\&=(-1)^nw_n^{2n+1}\int_0^\infty \varphi(w_nx)x^{2n}e^{ixz}\mathrm{d}x\end{aligned}$$
and using Ki's work (and including the error term) we have
$$f^{(2n)}(z/w_n)=\sqrt{\frac{\pi}{nw_n}}(-1)^ne^{-ae^{w_n}}e^{bw_n}w_n^{2n+1}e^{iz}\left(1+\mathcal{O}(w_n^{-2})\right).$$
From (3.4) we see that
$$\frac{1}{\Lambda^{2n}}\Xi_F^{(2n)}\left(\frac{z-M'}{\Lambda}\right)=Bf^{(2n)}(z)+\overline{B}f^{(2n)}(-z)$$
so setting $C_n=\frac{1}{\Lambda w_n}$,
$$\begin{aligned}(-1)^ne^{ae^{w_n}-bw_n}w_n^{-2n-1}\sqrt{\frac{nw_n}{\pi}}\frac{1}{|B|\Lambda^{2n}}\Xi_F^{(2n)}\left(C_nz-\frac{M'}{\Lambda}\right)&\\=\left(\frac{B}{|B|}e^{iz}+\frac{\overline{B}}{|B|}e^{-iz}\right)(1+\mathcal{O}(w_n^{-2}))&\\=2\cos(z+\arg(B))(1+\mathcal{O}(w_n^{-2}))&\end{aligned}$$
and after taking the limit, the proof Theorem 3.1 is complete. ☐

4. Numerical Demonstrations

In this section we briefly discuss how the L-function's data affects the convergence to the cosine function. Recall that the error term is $O(w_n^{-2})$ where

$$w_n \sim \log\left(\frac{2n}{a}\right),$$

with

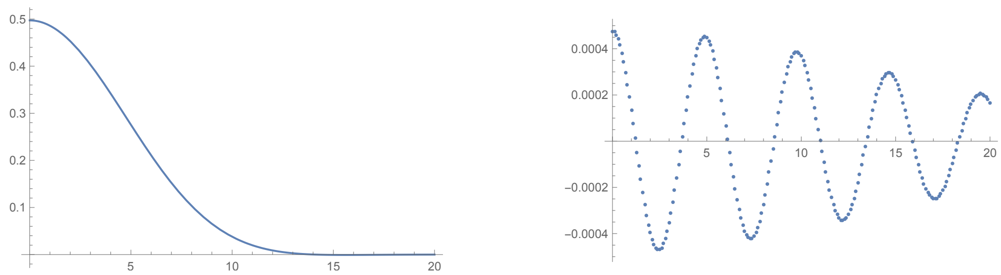
$$a = \Lambda Q^{-1/\Lambda} \prod_{j=1}^{k} \lambda_j^{-\lambda_j/\Lambda}.$$

Therefore L-functions with larger conductor converge slightly more quickly, and high degree L-functions converge more slowly. This fact is clearer if one assumes that one can transform the L-function so its data has $\lambda_j = 1/2$ for all j, since then $a = kQ^{-2/k}$.

The sequence C_n effectively scales the density of the zeros of the $(2n)^{\text{th}}$ derivative. We have that

$$C_n = \frac{1}{\Lambda w_n} \to 0.$$

which means that the zeros of the unscaled $(2n)^{\text{th}}$ derivative have moved towards the origin. Compare, for example, the Riemann Xi-function before any derivatives have been taken and after 100 derivatives have been taken.



(A) $n = 0$ (B) $n = 50$

Figure 4.1. Plots of the Riemann Xi-function after $2n$ derivatives

These figures also demonstrate the convergence to the cosine function.

Finally, the A_n term dictates how large the derivatives of the L-functions get. From

$$A_n = \frac{\sqrt{n}(-1)^n e^{ae^{w_n}} e^{-bw_n}}{2w_n^{2n+1/2}\sqrt{\pi}|B|\Lambda^{2n}}$$

and using the defining equation for w_n, $aw_n e^{w_n} = bw_n + 2n$, we have that

$$\log|A_n| = 2n(1 - \log\Lambda - \log w_n) - ae^{w_n}(w_n - 1) + \tfrac{1}{2}\log n - \tfrac{1}{2}\log w_n + O(1)$$

and so since $w_n \sim \log(2n/a)$, as $n \to \infty$ we have that $A_n \to 0$, which means that the size of the $(2n)^{\text{th}}$ derivative gets large as n increases, although for L-functions of small degree where $\log \Lambda < 1$ the size of the derivatives can initially decrease, before eventually increasing.

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